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Axiomatic resource allocation for heterogeneous agents

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Axiomatic resource allocation for heterogeneous agents

Juan D. MORENO-TERNERO [1] and John E. ROEMER[2]

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Abstract

We analyze a model of resource allocation in which agents' abilities (to transform the resource into an interpersonally comparable outcome) and initial endowments may differ. We impose ethical and operational axioms in this model and characterize some allocation rules as a result of combining these axioms. Two focal (and polar) egalitarian rules are singled out. On the one hand, the rule that allocates the resource equally across agents. On the other hand, the rule tha allocates the resource so that the distribution of final outcomes is lexicographically maximized.

Keywords: resource allocation, egalitarianism, priority, solidarity, composition.

JEL Classification: D63

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1 Introduction

Imagine the following basic problem. There is an amount of *wealth* to be allocated among individuals, each of whom possesses a capability to transform wealth into some given valued outcome, and the achievements of individuals, with regard to that outcome, are interpersonally comparable. Think, for instance, of life expectancy as a function of investment in health care, future earning power as a function of investment in education, etc.

In Moreno-Ternero and Roemer (2006) we analyze a canonical version of this problem in which no individual can achieve a positive level of the outcome without being allocated a positive amount of wealth. Individuals, however, are typically not only heterogeneous with respect to their capabilities, but also with respect to their initial endowments and/or opportunities. In this paper, we analyze a general model, encompassing the model in Moreno-Ternero and Roemer (2006), that also allows for heterogeneity of initial (outcome) endowments.

In resource allocation problems of this sort, there are usually two focal points of distribution: to distribute the available resource equally among all agents, and to distribute the resource among the population so as to equalize the outcomes among them. If agents differ in their initial endowments, this latter distribution might not be feasible.[1] The corresponding natural second best would be to distribute the resource among the population so as to lexicographically maximize the distribution of outcomes agents achieve after the allocation. We shall see that these two polar methods can actually be characterized together by combining several ethical and operational axioms. Among the former ones, there will be the so-called *priority* axiom (e.g., Moreno-Ternero and Roemer, 2006), which imposes a positive discrimination (but only to a certain extent) towards the less capable of transforming resource into outcome; and the so-called *resource monotonicity* axiom (e.g., Roemer, 1986) formalizing the idea of solidarity with respect to changes on the available wealth. Among the latter ones, there will be the so-called *composition* axiom (e.g., Young, 1988; Moulin, 2000) pertaining to the behavior of a rule with respect to tentative allocations based on a wrong estimation of the available wealth; and the *consistency* axiom (e.g., Thomson 2007) a principle of stability relating the solution of a given problem to the solutions of the corresponding reduced problems that appear when a subgroup of agents in the original population leaves with their awards.

[1] Think, simply, of the case in which only a small amount of wealth is available and some agents have large initial endowments, whereas most of the other agents have no initial endowment.

If the composition axiom is dropped from the characterization result mentioned above, more egalitarian rules are obtained. More precisely, a family of rules is obtained in which each rule lexicographically maximizes the distribution of some general *index* of resources and outcomes, where the index is not determined without further assumptions. This is actually a large class of rules having the focal resource-egalitarian and outcome-egalitarian rules described above as polar cases in that class, on the 'conservative' and 'radical' ends and generalizing the corresponding class of rules characterized in Moreno-Ternero and Roemer (2006).

Our paper can be considered as part of the fast-expanding literature on fair allocation. Traditionally, economists have been criticized for paying too little attention to distributional questions. There now exists, however, a well-developed literature devoted to the formulation and the analysis of equity concepts that traces back to Foley (1967) and his notion of envy-free allocation. In the last few years, a variety of new solutions has been proposed and applied to a wide range of models, and a number of properties of solutions have been formulated and studied for these models (see, for instance, Roemer, 1996b; Moulin, 2003; Fleurbaey, 2008; Thomson, forthcoming; and the literature cited therein). To a large extent, this literature has been axiomatic, taking as point of departure properties of allocation rules and investigating the existence of rules satisfying various combinations of these properties. This is precisely what we do in this paper for a model of resource allocation in which agents' capabilities and endowments may differ. Some of our axioms will reflect equity concerns and others will pertain to the way solutions respond to changes in some of the data describing the environment. As we shall see, our work will be a case in point for a broad notion of egalitarianism.

The use of the axiomatic method is not a discovery of the theory of fair allocation but of the theory of bargaining initiated by Nash (1950). Bargaining theory is the axiomatic study of utility-allocating mechanisms acting on a domain of utility possibility sets with threat points. The applications of bargaining theory to problems of distributive justice are usually not supportable, because too much justice-relevant information has been lost in the specification of the domain. This has motivated the extension of the theory to economic environments (e.g., Roemer, 1988). Our model is intimately connected to the resulting (extended) bargaining theory with economic environments.

2 The model

2.1 Preliminaries

Let $\mathbb{I}$ represent a population of agents (a set with an infinite number of members) who produce an objectively measurable outcome from a resource called wealth. For each $i \in \mathbb{I}$, let $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ be the individual function that transforms wealth into the outcome. We assume that, for each i, f_i is continuous, strictly increasing and unbounded. Note that $f_i(0)$ denotes the level of outcome that individual i achieves without being allocated any wealth whatsoever. We allow for individual heterogeneity by assuming that these levels (to be interpreted as initial endowments) may differ.[2]

Denote by $\mathcal{F}$ the set of functions satisfying the above properties, i.e.,

$$\mathcal{F} = \{f : \mathbb{R}_+ \to \mathbb{R}_+ : \text{continuous, strictly increasing and such that } \lim_{x\to\infty} f(x) = \infty\},$$

and by $\mathcal{F}^0$ the subset of these functions whose graphs emanate from the origin, i.e.,

$$\mathcal{F}^0 = \{f \in \mathcal{F} : f(0) = 0\}.$$

In other words, $\mathcal{F}^0$ is the set of outcome functions corresponding to those agents without initial endowments.

We shall also assume that $\{f_i : i \in \mathbb{I}\}$ constitutes a *covering domain*, i.e., the graphs of these functions cover the positive orthant.

We say that an individual i is *disabled* with respect to another individual j if the former always needs at least the same wealth than the latter one to reach the same level of the outcome, i.e., if $f_i \leq f_j$. Note that it might well be the case that two agents cannot be compared in terms of their relative ability.

Let $\mathcal{I}$ be the family of all finite subsets of $\mathbb{I}$. We define an **economy** e as a triple (N, f, W), where $N = \{i_1, i_2, ..., i_n\} \in \mathcal{I}$ is the set of agents, $f = (f_i)_{i\in N}$ is the profile of their outcome functions (defined as above), and $W \in \mathbb{R}_+$ represents the available wealth. The family of all economies is $\mathcal{E}$. The following two subfamilies of economies are worth defining:

$$\mathcal{E}^0 \equiv \{e = (N, f, W) \in \mathcal{E} : f_i \in \mathcal{F}^0 \text{ for all } i \in N\},$$

and

$$\mathcal{E}^c \equiv \{e = (N, f, W) \in \mathcal{E} : \text{ for all } i, j \in N, \text{ either } f_i \leq f_j \text{ or } f_j \leq f_i\}.$$

[2] In doing so, we provide an extension of the model in Moreno-Ternero and Roemer (2006).

In words, $\mathcal{E}^0$ is the subfamily of economies in which agents have no initial endowments, and $\mathcal{E}^c$ is the subfamily of economies in which agents can be completely ranked according to their relative ability.

An *allocation rule* is a function R that associates to each economy $e = (N, f, W) \in \mathcal{E}$ a unique point $R(e) = (R_i(e))_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} R_i(e) = W$. That is, an allocation rule indicates how to distribute the wealth available in an economy among its members.

Examples of rules are the following. First, the rule that awards each agent the same amount:

Resource-Egalitarian *rule (ER):* $ER_i(N, f, W) = \frac{W}{n}$.

An alternative to the resource-egalitarian rule is obtained by focusing on the levels of outcome agents achieve, as opposed to the resources they receive, and choosing the vector at which these outcome levels are as equal as possible.

Outcome-Egalitarian *rule (EO):* $EO_i(N, f, W) = \max\{f_i^{-1}(\lambda), 0\}$*, where* $\lambda > 0$ *is chosen so that* $\sum_{i \in N} \max\{f_i^{-1}(\lambda), 0\} = W$.

We shall present several characterization results for allocation rules. We begin by introducing our axiom of *priority*, which says that no agent can dominate another agent both in resources and outcome.

Priority *(PR). Let* $e = (N, f, W) \in \mathcal{E}$ *and* $i, j \in N$ *be such that* $R_i(e) < R_j(e)$*. Then* $f_i(R_i(e)) \geq f_j(R_j(e))$.

Note that this axiom (first introduced in Moreno-Ternero and Roemer (2006)) guarantees that disabled agents receive at least as much wealth as abler ones. In other words, priority implies the *weak equity* axiom, introduced by Sen (1973, 1974), and therefore a weak version of anonymity (usually referred as *symmetry*) which says that agents that are equally able are rewarded equally. On the other hand, priority also says that a disabled person is never resourced to the extent that her outcome level exceeds that of an able agent.[3]

Our next axiom says that, when a bad or good shock comes to an economy, all its members should share in the calamity or windfall. In doing so, we are capturing an instance of the idea of solidarity. This axiom has been previously used in related models by Roemer (1986) and Moulin and

[3]A similar notion to the one conveyed by the priority axiom is formalized in the context of compensation and responsibility by means of two dual axioms named acknowledged handicap and acknowledged merit, respectively (e.g., Fleurbaey, 2008).

Thomson (1988) among others. It is also reminiscent of the monotonicity axiom in bargaining theory (e.g., Kalai, 1977).

Resource monotonicity *(RM). Let $e = (N, f, W)$ and $e' = (N, f, W') \in \mathcal{E}$ be such that $W' < W$. Then $R_i(e') < R_i(e)$ for all $i \in N$ such that $R_i(e) > 0$ and $R_i(e') = R_i(e)$ for all $i \in N$ such that $R_i(e) = 0$.*

Our next principle, *consistency*, has played a fundamental role in axiomatic analysis (see, e.g., Thomson (2007) and the literature cited therein). It says that if a sub-group of two agents secedes with the resource allocated to it under F then, in the smaller economy, F allocates the resource in the same way. In that sense, consistency can be interpreted as a notion of stability.

Consistency *(CY). Let $e = (N, f, W) \in \mathcal{E}$. Let $N' \subset N$ with $|N'| = 2$ and $e' = (N', f', W')$, where $f' = (f_i)_{i \in N'}$ and $W' = \sum_{i \in N'} R_i(e)$. Then $R_i(e) = R_i(e')$, for all $i \in N'$.*

2.2 A family of rules

We now construct a family of allocation rules encompassing the examples presented above. To do so, let Φ be the family of all functions $\varphi : \mathbb{R}^2_+ \to \mathbb{R}_+$, continuous on its domain and non-decreasing, such that $\inf\{\varphi(x, y)\} = \varphi(0, 0) = 0$ and for all $(x, y) > (z, t)$, $\varphi(x, y) > \varphi(z, t)$.

Let φ be a function in the class Φ. For all $i \in \mathbb{I}$ define the function $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi_i(w) = \varphi(w, f_i(w))$ for all $w \in \mathbb{R}_+$. It is straightforward to show that, for all $i \in \mathbb{I}$, ψ_i is continuous and strictly increasing. For all $i \in \mathbb{I}$, let $\phi_i = (\psi_i)^{-1}$. Then, ϕ_i is continuous and strictly increasing. Then, define the $\varphi-$egalitarian rule, called L^φ, by

$$L_i^\varphi(N, f, W) = \max\{\phi_i(\xi), 0\}, \tag{1}$$

where $\xi > 0$ is chosen so that $\sum_{i \in N} \max\{\phi_i(\xi), 0\} = W$. This means that $L^\varphi(e)$ is the wealth allocation that lexicographically maximizes the φ-*value* across agents in e. Note that, applied in this manner to an agent's wealth and outcome, φ can be considered to be a generalized index of wealth and outcome. So the rules just defined leximin a generalized index of wealth and outcome. We show now that the L^φ rules actually equalize the index φ when restricted to economies in $\mathcal{E}^0$. More precisely,

Lemma 1 *For each $e = (N, f, W) \in \mathcal{E}^0$ and $\varphi \in \Phi$ denote $x = (x_i)_{i \in N} = L^\varphi(e)$. Then, $\varphi(x_i, f_i(x_i)) = \varphi(x_j, f_j(x_j))$ for all $i, j \in N$.*

Proof. Let $e = (N, f, W) \in \mathcal{E}^0$ and $\varphi \in \Phi$ be given and denote $x = (x_i)_{i \in N} = L^{\varphi}(e)$. Suppose, by contradiction, that $\varphi(x_i, f_i(x_i)) > \varphi(x_j, f_j(x_j))$ for some $i, j \in N$. Since $\inf\{\varphi(x, y)\} = \varphi(0, 0) = 0$, then $x_i > 0$. Thus, by continuity of φ, f_i and f_j, and taking into account that $f_i, f_j \in \mathcal{F}^0$, it follows that

$$\varphi(x_i - \varepsilon, f_i(x_i - \varepsilon)) > \varphi(x_j + \varepsilon, f_j(x_j + \varepsilon)),$$

for some $\varepsilon > 0$ sufficiently small. This contradicts the premise that we have leximinned φ. ■

It is straightforward to show that the resource-egalitarian rule and the outcome-egalitarian rule belong to the family of rules just described. Formally, $ER \equiv L^{\varphi_1}$, where $\varphi_1(x, y) = x$, and $EO \equiv L^{\varphi_2}$, where $\varphi_2(x, y) = y$.[4] These two rules are the extreme prioritarian rules for the most able and the least able agents in an economy where agents can be completely ranked according to their relative ability. More precisely, in such an economy, ER is the best (worst) prioritarian rule for the ablest (disablest) agent, whereas EW is the best (worst) prioritarian rule for the disablest (ablest) agent.

Proposition 1 *Let $e = (N, f, W) \in \mathcal{E}^c$. Assume that i and j are, respectively, the ablest and disablest agent in e. Then, for all rules (R) satisfying PR we have the following:*
(i) $ER_i(e) \geq R_i(e) \geq EO_i(e)$
(ii) $ER_j(e) \leq R_j(e) \leq EO_j(e)$

Proof. Let R be a rule satisfying PR. Let $e = (N, f, W) \in \mathcal{E}^c$ and let i (j) be the ablest (disablest) individual in e. We shall show only (i), as the proof of (ii) is analogous. Suppose, contrary to the claim, that $ER_i(e) < R_i(e)$. Then, there exists $k \in N$ such that $ER_k(e) > R_k(e)$. Since $ER_k(e) = ER_i(e)$, it follows that $R_i(e) > R_k(e)$, which contradicts the weak equity axiom, and therefore PR. Similarly, if $R_i(e) < EO_i(e)$, there exists $k \in N$ such that $EO_k(e) < R_k(e)$. Since f_i and f_k are strictly increasing, and $f_i(0) \geq f_k(0)$, it follows that $f_i(R_i(e)) < f_i(EO_i(e)) \leq f_k(EO_k(e)) < f_k(R_k(e))$. Then, by PR, $R_i(e) \geq R_k(e)$. However, since $f_i \geq f_k$ and f_i is strictly increasing, we have that $R_i(e) < R_k(e)$, which represents a contradiction. ■

In particular, Proposition 1 shows that, for all $\varphi \in \Phi$ and $e \in \mathcal{E}^c$,

$$ER_i(e) \geq L_i^{\varphi}(e) \geq EW_i(e) \text{ and } ER_j(e) \leq L_j^{\varphi}(e) \leq EW_j(e),$$

[4] Actually, increasing transformations of φ_1 and φ_2 would yield the same rules.

where i and j are, respectively, the ablest and disablest agents in e.

We can define a duality relationship between the members of the $\{L_\varphi\}_{\varphi\in\Phi}$ family as follows. For each $\varphi \in \Phi$, let φ^* be defined as $\varphi^*(x,y) = \varphi(y,x)$. Then, $\varphi^* \in \Phi$. We define the **dual** rule of L_φ as L_{φ^*}. L_φ and L_{φ^*} are symmetric with respect to the treatment of resources and outcome. Note that ER and EW are dual rules.

3 A characterization

We now show in this section that the family $\{L^\varphi\}_{\varphi\in\Phi}$ is characterized by the axioms introduced above. Formally,

Theorem 1 *The $\{L^\varphi\}_{\varphi\in\Phi}$ rules are characterized by RM, PR and CY.*

Proof. It is not difficult to show that the L^φ rules satisfy RM, CY and PR. Conversely, let R be a rule that satisfies RM, CY and PR.

Let $i \in \mathbb{I}$ and $\alpha \in \mathbb{R}_{++}$ be given. Define $E(R,i,\alpha) = \{e \in \mathcal{E} : R_i(e) = \alpha\}$ and $C(R,i,\alpha) = \{(a,b) \in \mathbb{R}^2_+ : a = R_j(e); b = f_j(a)$ for some $e = (N,f,W) \in E(R,i,\alpha)$ and $j \in N\}$. For ease of notation, let $\widehat{C}(R,i,\alpha) = C(R,i,\alpha) \cap \mathbb{R}^2_{++}$.[5] Our aim is to show that the family of curves $\{C(R,i,\alpha) : \alpha \in \mathbb{R}_{++}\}$ is the isoquant map of an appropriate function $\varphi \in \Phi$ and to show from here that $R = L^\varphi$.

We show first that any $C(R,i,\alpha)$ is *downward sloping*, i.e., if (a,b), $(a',b') \in C(R,i,\alpha)$ and $a' > a$ then $b' \leq b$. Suppose, to the contrary, that $b' > b$. By definition, there exist $e = (N,f,W)$ and $e' = (N',f',W') \in E(R,i,\alpha)$ and $j \in N$, $k \in N'$ such that $(a,b) = (R_j(e), f_j(R_j(e)))$ and $(a',b') = (R_k(e'), f_k(R_k(e')))$. Let $e^j = (\{i,j\},(f_i,f_j),\alpha+a)$ and $e^k = (\{i,k\},(f_i,f_k),\alpha+a')$. Then, by CY, $R(e^j) = (\alpha,a)$ and $R(e^k) = (\alpha,a')$. Let $e^* = (\{i,j,k\},(f_i,f_j,f_k),\alpha+a+a')$. Then, by CY and RM, $R(e^*) = (\alpha,a,a')$, which says, in particular, that $e^* \in E(R,i,\alpha)$ and $R_j(e^*) < R_k(e^*)$. Thus, PR implies that $f_j(R_j(e^*)) \geq f_k(R_k(e^*))$. However, our hypothesis says that $b = f_j(R_j(e^*)) < f_k(R_k(e^*)) = b'$, a contradiction.

We show now that $\{\widehat{C}(R,i,\alpha) : \alpha \in \mathbb{R}_{++}\}$ is a collection of disjoint sets. Suppose, by contradiction, that there exists $(a,b) \in \widehat{C}(R,i,\alpha_1) \cap \widehat{C}(R,i,\alpha_2)$. Assume, without loss of generality, that $\alpha_1 > \alpha_2 > 0$. Let

[5] Note that, for any $\alpha \in \mathbb{R}_{++}$, $C(R,i,\alpha) \cap \{x=0\} = \emptyset$, but it might well be the case that $C(R,i,\alpha) \cap \{y=0\} \neq \emptyset$, for some $\alpha \in \mathbb{R}_{++}$.

$e_1 = (N_1, f_1, \alpha_1) \in E(R, i, \alpha_1)$, $e_2 = (N_2, f_2, \alpha_2) \in E(R, i, \alpha_2)$ and $j \in N_1$, $k \in N_2$ such that $(a, b) = (R_j(e_1), f_j(R_j(e_1))) = (R_k(e_2), f_k(R_k(e_2)))$. Let $\widehat{e}_1 = (\{i, j\}, (f_i, f_j), a + \alpha_1)$ and $\widehat{e}_2 = (\{i, k\}, (f_i, f_k), a + \alpha_2)$. CY implies that $R_i(\widehat{e}_1) = \alpha_1$ and $R_i(\widehat{e}_2) = \alpha_2$. By PR and RM, there exists $W > a + \alpha_2$ for which $\widetilde{e}_2 = (\{i, k\}, (f_i, f_k), W) \in E(R, i, \alpha_1)$. By RM, applied to $\widehat{e}_2$ and $\widetilde{e}_2$, we know that $R_k(\widetilde{e}_2) > R_k(\widehat{e}_2) = a$. Therefore, $(a, b) < (R_k(\widetilde{e}_2), f_k(R_k(\widetilde{e}_2))) \in C(R, i, \alpha_1)$. This contradicts the fact that $C(R, i, \alpha_1)$ is downward sloping.

Next, we show that if $\alpha_1 > \alpha_2 > 0$ then $\widehat{C}(R, i, \alpha_1)$ *lies above* $\widehat{C}(R, i, \alpha_2)$, i.e.,

(i) For all $(a, b) \in \widehat{C}(R, i, \alpha_2)$ there exists $(a', b') \in \widehat{C}(R, i, \alpha_1)$ such that $(a, b) < (a', b')$.

(ii) There is no $(a'', b'') \in \widehat{C}(R, i, \alpha_2)$ and $(a, b) \in \widehat{C}(R, i, \alpha_1)$ such that $(a'', b'') < (a, b)$.

(i) Let $(a, b) \in \widehat{C}(R, i, \alpha_2)$. Then, there exists $j \in \mathbb{I}$ such that $f_j(a) = b$, for which $R(e) = (\alpha_2, a)$, where $e = (\{i, j\}, (f_i, f_j), \alpha_2 + a)$. By PR and RM, there exists $W^* > \alpha_2 + a$ for which $R_i(e^*) = \alpha_1$, where $e^* = (\{i, j\}, (f_i, f_j), W^*)$. Let $(a', b') = (R_j(e^*), f_j(R_j(e^*)))$. Then, $(a', b') \in \widehat{C}(R, i, \alpha_1)$. Furthermore, since F satisfies RM and f_j is strictly increasing, $(a', b') > (a, b)$.

(ii) Let $(a, b) \in \widehat{C}(R, i, \alpha_1)$. Suppose, by contradiction, that there exists $(a'', b'') \in \widehat{C}(R, i, \alpha_2)$ such that $(a'', b'') > (a, b)$. Then, by (i), there exists $(a''', b''') \in \widehat{C}(R, i, \alpha_1)$ such that $(a''', b''') > (a'', b'')$. Thus, $(a''', b''') > (a, b)$, which contradicts the fact that $C(R, i, \alpha_1)$ is downward sloping.

Finally, we show that if $(0, y) \in C(R, i, \alpha)$ then $(0, y') \in C(R, i, \alpha)$ for all $y' > y$.

Assume, by contradiction, that $(0, y) \in C(R, i, \alpha)$ and $(0, y') \notin C(R, i, \alpha)$ for some $y' > y$. Let $j \in \mathbb{I}$ be such that $f_j(0) = y'$ and $e = (\{i, j\}, (f_i, f_j), \alpha)$. Then, $R(e) \neq (\alpha, 0)$. In other words, $R(e) = (\alpha - \varepsilon, \varepsilon)$, for some $\varepsilon > 0$. Thus, $(\varepsilon, f_j(\varepsilon)) \in \widehat{C}(R, i, \alpha - \varepsilon)$. Now, let α_ε be such that $(\varepsilon/2, \alpha_\varepsilon) \in C(R, i, \alpha)$. By downward slopiness, $\alpha_\varepsilon \leq y < y' < f_j(\varepsilon)$. Since $\widehat{C}(R, i, \alpha)$ lies above $\widehat{C}(R, i, \alpha - \varepsilon)$, we have a contradiction.

Let i be the *identity* agent, i.e., i is such that $f_i(x) = x$ for all $x \in \mathbb{R}_+$. Let $(a, b) \in \mathbb{R}^2_{++}$. By the assumption of covering domain, there exists $j \in \mathbb{I}$ such that $f(a) = b$. Then, by RM and PR, there exists some $W > 0$ for which $R_j(e) = a$, where $e = (\{i, j\}, (f_i, f_j), W)$. Let $\alpha = W - a > 0$.[6] Then,

[6]Note that, by PR, the identity agent can never be awarded with zero resources.

$(a,b) \in \widehat{C}(R,i,\alpha)$. By the above, α is unique. Define then $\varphi : \mathbb{R}^2_{++} \to \mathbb{R}_+$ by $\varphi(a,b) = \alpha$, where $\alpha \in \mathbb{R}_{++}$ is the unique number for which $(a,b) \in \widehat{C}(R,i,\alpha)$.

For $y \in \mathbb{R}_+$, let $A(y) = \{\alpha \in \mathbb{R}_+ \text{ such that } (0,y) \in C(R,i,\alpha)\}$. Then, we extend the domain of φ by assuming that,

$$\varphi(0,y) = \begin{cases} 0 & \text{if } A(y) = \emptyset \\ \sup\{A(y)\} & \text{otherwise} \end{cases}$$

It is straightforward to show that $A(0) = \emptyset$, from where it follows that $\varphi(0,0) = 0 \leq \varphi(x,y)$ for all $(x,y) \in \mathbb{R}^2_+$.

Let $x, x', y \in \mathbb{R}_{++}$ such that $x < x'$. If $\varphi(x,y) > \varphi(x',y)$ then $\widehat{C}(R,i,\varphi(x,y))$ lies above $\widehat{C}(R,i,\varphi(x',y))$. In such a case, since $(x',y) \in \widehat{C}(R,i,\varphi(x',y))$, there exists $(z,t) \in \widehat{C}(R,i,\varphi(x,y))$ such that $(x',y) < (z,t)$. Then, $(z,t) > (x,y)$. Since $(x,y) \in \widehat{C}(R,i,\varphi(x,y))$, this contradicts that $C(R,i,\varphi(x,y))$ is downward sloping. Similarly, we show that $\varphi(x,y) \leq \varphi(x,y')$ for all $x, y, y' \in \mathbb{R}_{++}$ such that $y < y'$.

Let $x, y \in \mathbb{R}_{++}$ and assume, by contradiction, that $\varphi(0,y) > \varphi(x,y)$. Let $\widehat{\alpha} = \varphi(0,y)$. Since $C(R,i,\widehat{\alpha})$ is downward sloping, it follows that either $(x,y) \in \widehat{C}(R,i,\widehat{\alpha})$ or $(x,y) \in \widehat{C}(R,i,\alpha)$, for some $\alpha > \widehat{\alpha}$, which would imply either $\varphi(x,y) = \widehat{\alpha}$ or $\varphi(x,y) = \alpha$, a contradiction in any case.

Let $y < y' \in \mathbb{R}_{++}$. Then, if $(0,y) \in C(R,i,\alpha)$ so does $(0,y')$. Thus, $\varphi(0,y') \geq \varphi(0,y)$.

Let $(x,y), (z,t) \in \mathbb{R}^2_{++}$ such that $(x,y) > (z,t)$. By downward slopiness, $\varphi(x,y) \neq \varphi(z,t)$. If $\varphi(x,y) < \varphi(z,t)$ then $\widehat{C}(R,i,\varphi(z,t))$ lies above $\widehat{C}(R,i,\varphi(x,y))$. We have, however, that $(x,y) \in \widehat{C}(R,i,\varphi(x,y))$, $(z,t) \in \widehat{C}(R,i,\varphi(z,t))$ and $(x,y) > (z,t)$, which represents a contradiction.

Let $y, z, t \in \mathbb{R}_{++}$ such that $y < t$. As before, by downward slopiness, $\varphi(0,y) \neq \varphi(z,t)$. Assume, by contradiction, that $\varphi(0,y) > \varphi(z,t)$. Let $\alpha_1 = \varphi(0,y)$ and $\alpha_2 = \varphi(z,t)$. Then, $(z,t) \in \widehat{C}(R,i,\alpha_2)$ and, by downward slopiness, for some $\varepsilon > 0$, there exists $y_\varepsilon \leq y$ such that $(\varepsilon, y_\varepsilon) \in \widehat{C}(R,i,\alpha_1)$, which contradicts the fact that $\widehat{C}(R,i,\alpha_1)$ lies above $\widehat{C}(R,i,\alpha_2)$.

Finally, we show that φ is continuous on $\mathbb{R}^2_{++}$. Let $\{(a_n,b_n)\} \to (a,b) \in \mathbb{R}^2_{++}$. We must show that $\{\alpha_n\} = \{\varphi(a_n,b_n)\} \to \alpha = \varphi(a,b)$. If such is not the case, then there exists a subsequence $\{\alpha_{k_n}\}$ that converges to $\overline{\alpha} \neq \alpha$. We assume that $\overline{\alpha} < \alpha$.[7] Then, for k_n sufficiently large, $\alpha_{k_n} < \frac{\alpha+\overline{\alpha}}{2} < \alpha$, and therefore, $\widehat{C}(R,i,\alpha_{k_n})$ lies below $\widehat{C}(R,i,\frac{\alpha+\overline{\alpha}}{2})$ and this one lies below

[7] The proofs for the cases $\alpha < \overline{\alpha} < \infty$ and $\overline{\alpha} = \infty$ are similar.

$\widehat{C}(R,i,\alpha)$. In particular, there exists a ball B, about $(a,b) \in \widehat{C}(R,i,\alpha)$ which lies above $\widehat{C}(R,i,\frac{\alpha+\overline{\alpha}}{2})$. Since $(a_n,b_n) \to (a,b)$, it follows that for large k_n, $(a_{k_n},b_{k_n}) \in B$. On the other hand, $(a_{k_n},b_{k_n}) \in \widehat{C}(R,i,\alpha_{k_n})$ which, for large k_n, lies below $\widehat{C}(R,i,\frac{\alpha+\overline{\alpha}}{2})$. This represents a contradiction. Note that an analogous argument allows us to show the case in which $a=0$ but $(a_n,b_n) \in \mathbb{R}^2_{++}$. To conclude, assume that $a_n = a = 0$ for all n. If, by contradiction, there exists a subsequence $\{\alpha_{k_n}\}$ that converges to $\overline{\alpha} \neq \alpha$,[8] then, for k_n sufficiently large, $\alpha_{k_n} < \frac{\alpha+\overline{\alpha}}{2} < \alpha$. Then, $(0,b_{k_n}) \in C(R,i,\alpha_{k_n})$ whereas $(0,b_{k_n}) \notin C(R,i,\frac{\alpha+\overline{\alpha}}{2})$. Now, there is an interval centered in $(0,b)$ which lies within $C(R,i,\frac{\alpha+\overline{\alpha}}{2})$, which implies, by convergence, that $(0,b_{k_n}) \in C(R,i,\frac{\alpha+\overline{\alpha}}{2})$, a contradiction.

The proof of the theorem concludes by showing that $R = L^{\varphi}$, i.e., $R(N,f,W) = L^{\varphi}(N,f,W)$ for all $(N,f,W) \in \mathcal{E}$. Fix $e=(N,f,W) \in \mathcal{E}$. Let i be the *identity* agent, i.e., i is such that $f_i(x)=x$ for all $x \in \mathbb{R}_+$. Two cases are distinguished.

Case 1: $i \in N$. Let $\lambda = R_i(e) > 0$. Then, $(R_j(e), f_j(R_j(e))) \in C(R,i,\lambda)$ for all $j \in N$. Thus, $\psi_j(R_j(e)) = \lambda$ for all $j \in N$ such that $R_j(e) > 0$ and $\psi_j(R_j(e)) \geq \lambda$ for all $j \in N$ such that $R_j(e)=0$. Since $\sum_{j \in N} R_j(e) = W$, it follows that $R(e) = L^{\varphi}(e)$.

Case 2: $i \notin N$. Let $j \in N$ be such that $R_j(e) > 0$ and denote $w_j = R_j(e)$. By *PR* and *RM*, there exists $w_i > 0$ such that $R(\widehat{e}) = (w_i, w_j)$, where $\widehat{e} = (\{i,j\},(f_i,f_j),w_i+w_j)$. We show that $C(R,j,w_j) \subseteq C(R,i,w_i)$. To do so, let $(a,b) \in C(R,j,w_j)$. Then, there exists $l \in \mathbb{I}$ such that $b = f_l(a)$ and $(w_j,a) = (R_j(e^2),R_l(e^2))$, where $e^2 = (\{j,l\},(f_j,f_l),w_j+a)$. Then, by *CY* and *RM*, $R(e^3) = (w_i,w_j,a)$, where $e^3 = (\{i,j,l\},(f_i,f_j,f_l),w_i+w_j+a)$. Consequently, $(a,b) \in C(R,i,w_i)$, showing that $C(R,j,w_j) \subseteq C(R,i,w_i)$. We now distinguish two subcases.

Subcase 2.1: $R_l(e) = 0$ for all $l \in N \backslash \{j\}$. In other words, $W = w_j$. Since $(0,f_l(0)) = (R_l(e), f_l(R_l(e))) \in C(R,j,w_j)$ for all $l \in N \setminus \{j\}$, the above shows that $(R_l(e), f_l(R_l(e))) \in C(R,i,w_i)$ for all $l \in N$. Thus, $\psi_l(0) \geq w_i$ for all $l \in N \backslash \{j\}$ and $\psi_j(W) = w_i$. From here, the proof of Case 1 concludes

Subcase 2.2: $R_k(e) > 0$ for some $k \in N \setminus \{j\}$. Denote $w_k = R_k(e) > 0$. As above, by *PR* and *RM*, there exists $\widetilde{w}_i > 0$ such that $R(\widetilde{e}) = (\widetilde{w_i}, w_k)$, where $\widetilde{e} = (\{i,k\},(f_i,f_k),\widetilde{w}_i+w_k)$. Analogously to the above argument, we show that $C(R,k,w_k) \subseteq C(R,i,\widetilde{w}_i)$. Thus, it follows that $(w_j,f_j(w_j)) \in \widehat{C}(R,i,w_i) \cap \widehat{C}(R,i,\widetilde{w}_i)$, which implies that $w_i = \widetilde{w}_i$. Let $\lambda = w_i = \widetilde{w}_i$.

[8] We assume, as before, that $\overline{\alpha} < \alpha$.

Then, $(R_l(e), f_l(R_l(e))) \in C(R, i, \lambda)$ for all $l \in N$. From here, the proof of Case 1 concludes. ■

Lemma 1 shows that Theorem 1 is indeed a generalization of Theorem 1 in Moreno-Ternero and Roemer (2006), where we show that, for the domain of economies $\mathcal{E}^0$, the family of rules that equalize an index of resources and outcome levels is characterized by the corresponding axiom of restricted domain, the axiom of priority and a solidarity axiom (which is equivalent in that context to the combination of resource monotonicity and consistency).

4 A new axiom and a new characterization

Next, we introduce a property pertaining to the behavior of a rule with respect to tentative allocations based on a wrong estimation of the available wealth. To motivate this property, imagine the following scenario: after having committed to divide the available wealth, one finds that the actual amount to divide is larger than was initially assumed. Then, two options are open: either the tentative division is cancelled altogether and the allocation for the actual economy is obtained directly, or we add to the initial commitment the result of applying the rule to the subsequent economy with the remaining amount and the adjusted individual outcome functions that would emerge after the initial economies. The requirement of *composition* is that both ways of proceeding should result in the same allocations. Formally,

Composition *(CP). Let* $e = (N, f, W) \in \mathcal{E}$. *Let* $W^1, W^2 \in \mathbb{R}_{++}$ *such that* $W = W^1 + W^2$ *and* $e^1 = (N, f, W^1) \in \mathcal{E}$. *For each* $i \in N$, *let* $\widehat{f}_i \in \mathcal{F}$ *be such that* $\widehat{f}_i(x) = f_i(x + R_i(e^1))$ *for all* $x \in \mathbb{R}_+$, *and let* $e^2 = \left(N, \left(\widehat{f}_i\right)_{i \in N}, W^2\right) \in \mathcal{E}$. *Then,* $R(e) = R(e^1) + R(e^2)$.[9]

The property of composition has been frequently used in the context of rationing problems (e.g., Young, 1988; Moulin, 2000). It is also reminiscent of the step-by-step negotiations axiom in bargaining (e.g., Kalai, 1977; Myerson, 1977). Similar notions have also appeared in alternative contexts (e.g., Laffond et al., 1996; Moulin and Stong, 2002).

It is straightforward to show that ER satisfies CP. We also have the following.

Proposition 2 *EO satisfies CP.*

[9]Note that CP implies a weak version of RM, but not necessarily the strict version we are using here.

Proof. Let $e = (N, f, W) \in \mathcal{E}$. Then, for each $i \in N$, $EO_i(e) = \max\{\sigma_i(\lambda), 0\}$, where $\sigma_i = f_i^{-1}$ and λ is such that $\sum_{i \in N} \max\{\sigma_i(\lambda), 0\} = W$. Assume, without loss of generality, that $N = \{1, 2, ..., n\}$ and that agents are ranked (in an increasing order) according to their initial endowments, i.e., $f_i(0) \leq f_{i+1}(0)$ for all $i = 1, ..., n-1$. Let $W^1, W^2 \in \mathbb{R}_{++}$ be such that $W = W^1 + W^2$ and let $e^1 = (N, f, W^1) \in \mathcal{E}$. For each $i \in N$, let $\widehat{f}_i \in \mathcal{F}$ be such that $\widehat{f}_i(x) = f_i(x + EO_i(e^1))$ for all $x \in \mathbb{R}_+$, and let $e^2 = (N, \widehat{f}, W^2) \in \mathcal{E}$, where $\widehat{f} = (\widehat{f}_i)_{i \in N}$. Let $\widehat{\sigma}_i = \widehat{f}_i^{-1}$ for each $i \in N$. Then,

$$EO_i(e) = \begin{cases} \sigma_i(\lambda) & \text{for all } i = 1, \dots, k \\ 0 & \text{for all } i = k+1, \dots, n \end{cases}$$

where λ and k are such that

$$\sum_{i=1}^{k} \sigma_i(\lambda) = W, \text{ and } f_{k+1}(0) > \lambda \geq f_k(0).$$

Similarly,

$$EO_i(e^1) = \begin{cases} \sigma_i(\lambda_1) & \text{for all } i = 1, \dots, k_1 \\ 0 & \text{for all } i = k_1+1, \dots, n \end{cases}$$

where λ_1 and k_1 are such that

$$\sum_{i=1}^{k_1} \sigma_i(\lambda_1) = W^1, \text{ and } f_{k_1+1}(0) > \lambda_1 \geq f_{k_1}(0).$$

Thus, note that $k \geq k_1$ and $\lambda \geq \lambda_1$. Finally,

$$EO_i(e^2) = \begin{cases} \widehat{\sigma}_i(\lambda_2) & \text{for all } i = 1, \dots, k_2 \\ 0 & \text{for all } i = k_2+1, \dots, n \end{cases}$$

where λ_2 and k_2 are such that

$$\sum_{i=1}^{k_2} \widehat{\sigma}_{k_2}(\lambda_2) = W^2, \text{ and } \widehat{f}_{k_2+1}(0) > \lambda_2 \geq \widehat{f}_{k_2}(0).$$

Let $y = EO(e) - EO(e^1)$ and $z = EO(e^2)$. Let $\widehat{y} = \left(\widehat{f}_i(y_i)\right)_{i \in N}$ and $\widehat{z} = \left(\widehat{f}_i(z_i)\right)_{i \in N}$. Then, it is straightforward to show that

$$\widehat{y}_i = \begin{cases} \lambda & \text{for all } i = 1, \dots, k \\ f_i(0) & \text{for all } i = k+1, \dots, n \end{cases}$$

and

$$\widehat{z}_i = \begin{cases} \lambda_2 & \text{for all } i = 1, \ldots, k_2 \\ \widehat{f}_i(0) & \text{for all } i = k_2 + 1, \ldots, n \end{cases}$$

We have to show that $y = z$.[10] Note first that, since y is a feasible allocation for the economy e^2, it follows, by definition of EO, that $\widehat{z}$ lexicographically dominates $\widehat{y}$. This implies that $\lambda \leq \lambda_2$.[11] Thus, $k_1 \leq k \leq k_2$.[12] Then,

$$y_i = \begin{cases} \sigma_i(\lambda) - \sigma_i(\lambda_1) & \text{for all } i = 1, \ldots, k_1 \\ \sigma_i(\lambda) & \text{for all } i = k_1 + 1, \ldots, k \\ 0 & \text{for all } i = k + 1, \ldots, n \end{cases}$$

and

$$z_i = \begin{cases} \widehat{\sigma}_i(\lambda_2) & \text{for all } i = 1, \ldots, k_2 \\ 0 & \text{for all } i = k_2 + 1, \ldots, n \end{cases}$$

Let $i = 1, \ldots, k_1$. Then, $\widehat{f}_i(x) = R_i(x + \sigma_i(\lambda_1))$ for all $x \in \mathbb{R}_+$. Thus, $\widehat{\sigma}_i(x) = \sigma_i(x) - \sigma_i(\lambda_1)$ for all $x \in \mathbb{R}_+$. In particular, $\widehat{\sigma}_i(\lambda_2) = \sigma_i(\lambda_2) - \sigma_i(\lambda_1)$. Similarly, $\widehat{\sigma}_i(\lambda_2) = \sigma_i(\lambda_2)$ for all $i = k_1+1, \ldots, k_2$. Thus, $z_i \geq y_i$ for all $i \in N$. Now, if $\lambda < \lambda_2$, we would have $W^2 = \sum_{i \in N} z_i > \sum_{i \in N} y_i = W^2$, a contradiction. Thus, it follows that $\lambda = \lambda_2$ and, therefore that $k = k_2$, which implies that $y = z$, as desired. ■

Indeed, as the next result shows, ER and EO are characterized by CP, when this axiom is combined with the axioms introduced in the previous sections.

Theorem 2 *ER and EO are characterized by RM, PR, CY and CP.*

Proof. We already know one implication. In order to prove the converse one, note that, by Theorem 1, a rule that satisfies RM, PR, CY and CP is a member of the family of egalitarian rules L^{φ}. We show next that no other rule within this family, different from ER and EO, satisfies CP.

[10] Note that

$$\widehat{f}_i(0) = \begin{cases} \lambda_1 & \text{for all } i = 1, \ldots, k_1 \\ f_i(0) & \text{for all } i = k_1 + 1, \ldots, n \end{cases}$$

[11] Note that $\widehat{z}$ and $\widehat{y}$ have their coordinates increasingly ordered.

[12] Otherwise, if $k_1 \leq k_2 < k$, let $i = k_2 + 1 < k$. Then, $\widehat{y}_i = \lambda$ and, therefore, $f_i(0) \leq \widehat{y}_i \leq \lambda_2$. However, $f_i(0) = \widehat{f}_i(0) > \lambda_2$, a contradiction. Similarly, if $k_2 < k_1 \leq k$, let $i = k_2 + 1 < k_1 \leq k$. Then, $f_i(0) \leq \lambda$ and $\widehat{f}_i(0) > \lambda_2$, which represents a contradiction too, as $\widehat{f}_i(0) = \max\{\lambda_1, f_i(0)\} \leq \lambda$.

Let $\varphi \in \Phi \setminus \{\varphi_1, \varphi_2\}$, where $\varphi_1(x,y) = x$ and $\varphi_2(x,y) = y$ for all $(x,y) \in \mathbb{R}^2_+$.[13] Then, there exists $(x_0, y_0) \in \mathbb{R}^2_{++}$ such that $\varphi(x_0', y_0) < \varphi(x_0, y_0)$ for all $x_0' < x_0$. We distinguish two cases.

Case 1. $\varphi(0, y) > 0$ for some $y \in \mathbb{R}_+$.

Let y_1 be such that there exists some $\delta > 0$ for which $\varphi(0, y_1) = \varphi(x_0 - \delta, y_0) > 0$. Let $f_1 : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $f_1(x) = \frac{y_0}{x_0} x$ for all $x \in \mathbb{R}_+$ and $f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $f_2(x) = \frac{y_0}{x_0} x + y_1$ for all $x \in \mathbb{R}_+$. Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varphi(\varepsilon, \frac{y_0}{x_0}\varepsilon) < \varphi(0, y_1)$ and consider the economy $e^1 = (\{1,2\}, (f_1, f_2), \varepsilon)$. Then, it is straightforward to show that

$$L^{\varphi}(e^1) = (\varepsilon, 0).$$

For $i = 1, 2$, define the function $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ that determines the $\varphi-$value agent i achieves, depending on the wealth she receives, i.e., $\psi_i(w) = \varphi(w, f_i(w))$ for all $w \in \mathbb{R}_+$. Let $\lambda = \psi_1(x_0) = \varphi(x_0, y_0) > 0$ and $W = \psi_1^{-1}(\lambda) + \psi_2^{-1}(\lambda)$. Then, it is straightforward to show that

$$L^{\varphi}(e) = \left(\psi_1^{-1}(\lambda), \psi_2^{-1}(\lambda)\right).$$

Finally, let $W_2 = W - \varepsilon > 0$.[14] For $i = 1, 2$, let $\widehat{f}_i : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\widehat{f}_i(x) = f_i(x + L_i^{\varphi}(e^1))$ and consider the economy $e^2 = (\{1,2\}, (\widehat{f}_1, \widehat{f}_2), W^2)$. For $i = 1, 2$, let $\hat{\psi}_i : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\hat{\psi}_i(w) = \varphi(w, \widehat{f}_i(w))$ for all $w \in \mathbb{R}_+$. We distinguish two subcases.

Subcase 1.1. $L^{\varphi}(e^2) = (W_2, 0)$.

If so, $L^{\varphi}(e) = L^{\varphi}(e^1) + L^{\varphi}(e^2)$ if and only if $\psi_2^{-1}(\lambda) = 0$, which would imply $\varphi(x_0, y_0) = \lambda = \varphi(0, y_1) = \varphi(x_0 - \delta, y_0)$, a contradiction.

Subcase 1.2. $L^{\varphi}(e^2) = \left(\hat{\psi}_1^{-1}(\lambda'), \hat{\psi}_2^{-1}(\lambda')\right)$.

If so, $L^{\varphi}(e) = L^{\varphi}(e^1) + L^{\varphi}(e^2)$ if and only if $\psi_2^{-1}(\lambda) = \hat{\psi}_2^{-1}(\lambda')$ and $\psi_1^{-1}(\lambda) = \varepsilon + \hat{\psi}_1^{-1}(\lambda')$. From the former equality, it follows that $\lambda = \lambda'$, as $\hat{\psi}_2 \equiv \psi_2$ is a strictly increasing function. Thus, from the latter equality, it follows that $x_0 = \varepsilon + \hat{\psi}_1^{-1}(\lambda)$, or equivalently, $\varphi(x_0, y_0) = \lambda = \hat{\psi}_1(x_0 - \varepsilon) = \varphi(x_0 - \varepsilon, \widehat{f}_1(x_0 - \varepsilon)) = \varphi(x_0 - \varepsilon, f_1(x_0)) = \varphi(x_0 - \varepsilon, y_0)$, a contradiction.

Case 2. $\varphi(0, y) = 0$ for all $y \in \mathbb{R}_+$.

[13] Actually, increasing transformations of φ_1 and φ_2 would yield the same rules. Recall that $L^{\varphi_1} \equiv L^{\widehat{\varphi}_1} \equiv ER$ and $L^{\varphi_2} \equiv L^{\widehat{\varphi}_1} \equiv EO$, for any $\widehat{\varphi}_i$ increasing transformation of φ_i for $i = 1, 2$.

[14] Note that ε can be small enough.

Since $\varphi \in \Phi \setminus \{\varphi_1, \varphi_2\}$, we know that there exist $\delta_1, \delta_2 > 0$ such that $\varphi(x_0 + \delta_1, y_0 - \delta_2) = \varphi(x_0, y_0)$. It is also straightforward to show that there exists $\alpha > 0$ for which $\varphi(x_0 + \delta_1 - \alpha, y_0 - \delta_2) \neq \varphi(x_0 - \alpha, y_0)$.

Let $f_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be such that

- $f_1(x_0) = y_0$
- $f_2(x_0 + \delta_1) = y_0 - \delta_2$
- $f_1(\alpha) = f_2(\alpha)$

Let $W \in \mathbb{R}_+$ be such that $W > 2\alpha$. Then, consider the economies $e^1 = (\{1,2\}, (f_1, f_2), 2\alpha)$ and $e = (\{1,2\}, (f_1, f_2), W)$. It is straightforward to show that

$$L^{\varphi}(e^1) = (\alpha, \alpha),$$

and

$$L^{\varphi}(e) = \left(\psi_1^{-1}(\lambda), \psi_2^{-1}(\lambda)\right),$$

where λ is such that $\psi_1^{-1}(\lambda) + \psi_2^{-1}(\lambda) = W$. Equivalently,

$$L^{\varphi}(e) = (x, W - x)\,,$$

where

$$\varphi(x, f_1(x)) = \varphi(W - x, f_2(W - x)). \tag{2}$$

For $i = 1, 2$, let $\widehat{f}_i : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\widehat{f}_i(x) = f_i(x + \alpha)$ and consider the economy $e^2 = (\{1,2\}, (\widehat{f}_1, \widehat{f}_2), W^2)$, where $W^2 = W - 2\alpha$. For $i = 1, 2$, let $\hat{\psi}_i : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\hat{\psi}_i(w) = \varphi(w, \widehat{f}_i(w))$ for all $w \in \mathbb{R}_+$. Then,

$$L^{\varphi}(e^2) = \left(\hat{\psi}_1^{-1}(\lambda'), \hat{\psi}_2^{-1}(\lambda')\right),$$

where λ' is such that $\psi_1^{-1}(\lambda') + \psi_2^{-1}(\lambda') = W^2$. Equivalently,

$$L^{\varphi}(e^2) = (y, W - 2\alpha - y)\,,$$

where

$$\varphi(y, f_1(y + \alpha)) = \varphi(W - 2\alpha - y, f_2(W - \alpha - y)).$$

Then, $L^{\varphi}(e) = L^{\varphi}(e^1) + L^{\varphi}(e^2)$ if and only if $\alpha + y = x$. Or, equivalently, if and only if $\psi_1^{-1}(\lambda) = \alpha + \hat{\psi}_1^{-1}(\lambda')$. If so,

$$\varphi(x - \alpha, f_1(x)) = \varphi(W - x - \alpha, f_2(W - x)). \tag{3}$$

Now, if $W = 2x_0 + \delta_1$, then (2) becomes

$$\varphi(x, f_1(x)) = \varphi(2x_0 + \delta_1 - x, f_2(2x_0 + \delta_1 - x)),$$

which implies $x = x_0$.[15] Thus, (3) becomes

$$\varphi(x_0 - \alpha, y_0) = \varphi(x_0 + \delta_1 - \alpha, y_0 - \delta_2),$$

which represents a contradiction.[16]

5 Further insights

We have analyzed a simple distribution problem in which a given amount of wealth has to be distributed among individuals possessing a capability to transform wealth into some given valued (interpersonally comparable) outcome and, possibly, individual (outcome) endowments. For this simple environment, we have characterized the two focal egalitarian allocation rules that exist, i.e., the one that allocates the resource equally, and the one that makes outcome levels as equal as possible. We have shown that these two focal rules are the only *ethical* and *operational* procedures for allocating wealth, provided we assume that *ethical* means *prioritarian* and *solidaristic*, and *operational* means obeying the axioms of *composition* and *consistency*. Our characterization result therefore shows that the combination of the notions of priority, resource monotonicity, composition, and consistency is equivalent to a kind of egalitarianism, where the equality in question is either resources or outcome levels.

We have also shown that if we drop the axiom of composition in the result just described, then a whole family of egalitarian rules (somehow lying between the resource-egalitarian rule and the outcome-egalitarian rule) emerges. More precisely, the combination of the notions of priority, resource monotonicity, and consistency is equivalent to a kind of egalitarianism, where the equality in question is now some general *index* of resources and outcome levels. This result generalizes the result we obtain in Moreno-Ternero and Roemer (2006), for a simpler model in which agents are not allowed to be heterogeneous with respect to their initial endowments.

Discussions in moral philosophy have offered us a wide menu in answer to the question: equality of what? (e.g., Sen, 1980). In other words, if one

[15] Recall that $\varphi(x_0, y_0) = \varphi(x_0 + \delta_1, y_0 - \delta_2)$.

[16] It is worth noting that this argument requires $W = 2x_0 + \delta_1 > 2\alpha$. Now, we know that $x_0 - \alpha \geq 0$ and $x_0 + \delta_1 - \alpha > 0$, which guarantees that $2x_0 + \delta_1 > x_0 + \alpha \geq 2\alpha$.

is an egalitarian, what should one wish to equalize? Two well-known (and polar) theories arise to answer this question: the so-called *resource egalitarianism* and *welfare egalitarianism.* Our paper somehow builds a bridge between the two sides upon providing rationale for both egalitarian theories as well as a combination of them (exemplified by the family of egalitarian rules we obtain as a result of equalizing, as much as possible, an index of resources and outcome levels). Nevertheless, it is worth mentioning that ours is not a *welfarist* approach, the approach usually linked to welfare egalitarianism and which maintains that the justness of a state should be a function only of the welfares, or outcomes, of the agents in that state. We take instead a *resourcist* approach by studying allocation mechanisms defined on a space of economic environments, where the distribution of resources can be explicitly defined. In other words, we endorse the view that information concerning the distribution of goods or resources is in general necessary to evaluate the justness of a state of the world. As we have seen, this does not preclude us from obtaining welfare (in our case, outcome) egalitarianism as a result of combining some (non-welfarist) axioms. In a similar vein, welfare egalitarianism has also been characterized in models analyzing economic environments (e.g., Roemer, 1986; Sprumont, 1996; Chen and Maskin, 1999; Ginés and Marhuenda, 2000; Maniquet and Sprumont, 2005).[17]

ln moral and political philosophy, the debate between opportunity or resource egalitarians and outcome or welfare egalitarians is between those who wish to hold people responsible for the choices they make and preferences they have, after some initial equality has been guaranteed, and those who wish to hold individuals responsible for nothing about themselves (e.g., Roemer, 1986). A natural extension of the model in this paper would account for individual effort decisions. As a matter of fact, most (if not all) of the ethical axioms we use in this work would only be justified for equally-deserving individuals. The literature on compensation and responsibility, formally initiated by Bossert (1995) and Fleurbaey (1995) among others, provides us with an appropriate framework for such a natural extension.[18] In its simplest case, this literature deals with the allocation of a given amount of an external one-dimensional resource (which is not produced) among a group of individuals whose outcome achievements depend on this resource, but also on their *social background* (a characteristic which elicits compensation) and

[17] Needless to say that welfare egalitarianism has also been characterized in paradigmatic welfarist theories such as bargaining theory (e.g., Kalai, 1977; Myerson, 1977; Thomson, 1983).

[18] See Fleurbaey (2008) or Fleurbaey and Maniquet (forthcoming) for recent surveys on this literature.

personal effort (a characteristic which does not elicit compensation). In the parlance of our paper, this could be interpreted as saying that individual outcome functions would be bivariate functions depending on two variables reflecting the personal effort of the individual and the amount of the resource she is allocated. The mappings themselves would incorporate the influence of social background on individual outcome achievements. Characterizations of allocation rules for this model exist in the literature (e.g., Fleurbaey, 2008; Fleurbaey and Maniquet, forthcoming).[19] Therefore it would be interesting to explore whether the translation of our axioms to this context would give rise to new characterizations.[20]

Eventually, a theory of distributive justice must, we believe, postulate a domain of economies in which effort choices by individuals (relating to education and production), as well as risk preferences and level-comparable welfare, in a multi-stage model, are described. The present analysis is a far cry from that goal. Indeed, one difficulty in the work of philosophers is that they implicitly assume all these attributes of real-world societies in their theorizing. In any case, it is not risky to say that it would be immensely difficult to deduce formally a theory of just resource allocation on such a domain, without postulating unacceptably strong axioms, and so it is not surprising that the work of political philosophers is tentative and sketchy, by their own admission.[21]

[19] In order to address contexts where, due to incentive constraints, allocation rules cannot be implemented, attention has also been paid to social ordering functions for this kind of problems (e.g., Maniquet and Sprumont, 2005; Fleurbaey and Maniquet, 2006; Valletta, 2007).

[20] It is worth mentioning that the model we present in this paper can also accommodate a more general distribution problem, which could be considered an intermediate case between our benchmark analysis and the simplest model on compensation and responsibility described above. More precisely, think of the case in which agents achieve an interpersonally comparable outcome (which could be interpreted as *welfare*) as a function of the amount of an external one-dimensional resource they are allocated, called *wealth*, and, also, the level of a non-tradeable good, called *health*, they enjoy. In such a case, any individual would be described by a *trait* denoting her level of health and her level of (allocated) wealth. If the domain of individual traits is endowed with a complete and *natural* order, then a function representing this order would allow us to associate with each agent an individual outcome function indicating the welfare this agent achieves, after the allocation of wealth takes place. The axioms and results in this paper would carry over this alternative model.

[21] Roemer (1996a, Chapter 11) discusses how, when one admits more information into the description of environments which form the domain of a set of allocation rules, even stronger universal domain axioms are required to preserve simple characterizations of specific allocation rules. Indeed, when environments contain a great deal information about individuals, the required domain axiom becomes incredible. The mathematics is

A final comment is worth stressing. The literature on fair allocation has been traditionally opposed in its informational setting to the (polar) branch of welfare economics dealing with social welfare functions. Whereas the arguments of the social welfare functions are interpersonally comparable utilities, most of the models in fair allocation display only individual non-comparable preferences, and the equity requirements are all formulated in terms of individual preferences. This is no longer an ubiquitous feature in this literature, as there are also recent models in fair allocation assuming interpersonally comparable utilities (e.g., Bossert, 1995; Fleurbaey, 1995; 2008; Fleurbaey and Maniquet, forthcoming), as we do in of this paper. Nevertheless, the solutions that arise do not necessary involve interpersonal comparisons of utility, as with the equal-resource rule in our model.[22] The notion of relative disability, and the ensuing axiom of priority we consider, are formulated in terms of the (interpersonally comparable) outcome levels that individuals may achieve. In a more general context, such as the literature on compensation and responsibility described above, disability issues could be addressed without interpersonally comparable utilities, by means of ordinal preferences on extended bundles (i.e., external resources and effort levels). We do not endorse the view that social choice should study exclusively models which assume interpersonal comparability of welfare, although we think such models are important, especially when we interpret the individual outcome functions as ones that relate resources consumed to the degree to which various capabilities, objectively measurable, are realized. This has been our assumption in this article.

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telling us that there are no simple characterization rules on classes of environments that contained detailed descriptions of individuals, which is why political philosophy is generally quite indeterminate in its prescriptions.

[22] In the literature on compensation and responsibility this feature is actually a result of the so-called liberal reward axioms (e.g., Fleurbaey, 2008).

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